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Some considerations on topologies of infinite dimensional unitary coadjoint orbits

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Abstract

The topology of the embedding of the coadjoint orbits of the unitary group $\mathfrak{U}$ of an infinite dimensional complex Hilbert space $\mathcal{H}$, as canonically determined subsets of the space $\mathfrak{T}_s$ of symmetric trace-class operators, is investigated. The space $\mathfrak{T}_s$ is identified with the B-space predual of the Lie-algebra $\mathcal{L}(\mathcal{H})_s$ of the Lie group $\mathfrak{U}$. It is proved, that the orbits consisting of symmetric operators with finite range are (regularly embedded) closed submanifolds of $\mathfrak{T}_s$. Such orbits play a role of "generalized phase spaces" of (also nonlinear) quantum mechanics.

An alternative method of proving the regularity of the embedding is also given for the "one-dimensional" orbit, i.e. for the projective Hilbert space $P(\mathcal{H})$. Closeness of all the orbits lying in $\mathfrak{T}_s$ is also proved.

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1. Introduction

The aim of this section is a description of motivation, a formulation of the problem, and a brief discussion of the role played in physics by mathematical objects connected with it. The solution of the problem is presented in Theorem 2.5. Section 3 contains a proof of closeness of *all* unitary coadjoint orbits lying in the space $\mathfrak{T}_s$ of symmetric trace-class operators, as well as an alternative proof of Theorem 2.5 for a specific case.

Mathematical formalism of quantum mechanics (QM) is traditionally based on separable complex Hilbert space $\mathcal{H}$, and on closely connected objects: the W^*-algebra of bounded

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operators $\mathcal{L}(\mathcal{H})$, the $\sigma(\mathcal{L}(\mathcal{H}), \mathcal{L}(\mathcal{H})_*)$-continuous (with $\mathcal{L}(\mathcal{H})_* := \mathfrak{T} := L^1(\mathcal{H}) :=$ the trace-class operators on $\mathcal{H}$) linear functionals ν on $\mathcal{L}(\mathcal{H})$ (identified with $\nu \in \mathfrak{T}$ by $\nu(B) := \mathrm{Tr}(\nu B)\, \forall B \in \mathcal{L}(\mathcal{H})$), and the group of *-automorphisms $^*\text{-Aut}(\mathcal{L}(\mathcal{H}))$ of $\mathcal{L}(\mathcal{H})$ (acting on linear functionals by the transposed maps). Since each *-automorphism $\alpha \in {}^*\text{-Aut}(\mathcal{L}(\mathcal{H}))$ of the W^*-algebra $\mathcal{L}(\mathcal{H})$ is inner, it is described by a unitary operator ($\mathfrak{U}$ is the set of all unitary elements of $\mathcal{L}(\mathcal{H})$) $u_\alpha \in \mathfrak{U} : \alpha(B) \equiv u_\alpha B u_\alpha^*$ (u_α is determined by α up to a numerical factor). The transposed maps are then described by $\nu \mapsto u_\alpha^* \nu u_\alpha$.

The bounded symmetric operators composing $\mathcal{L}(\mathcal{H})_s$ are interpreted in QM as "observables", the *-automorphisms of $\mathcal{L}(\mathcal{H})$ are "symmetries", and the positive elements $\rho \in \mathfrak{T}_s \subset \mathfrak{T}$ with $\mathrm{Tr}(\rho) = 1$ are "density matrices" describing "states" of a physical system. Symmetries (and dynamics, as a specific one-parameter group case) of these QM systems are described equivalently also by actions of the mentioned transposed maps leaving the set $\mathfrak{T}_s$ invariant. This rough picture of physically interpreted transformations extends also to a natural nonlinear extension of QM described in [1] and extending the formalism to much wider class of dynamical systems. The orbits of such actions are always contained in orbits $\mathcal{O}_\rho(\mathfrak{U}) := \{u^* \rho u : u \in \mathfrak{U}\}$ of the action of the whole $\mathfrak{U}$ on any $\rho \in \mathfrak{T}_s$; the $\mathcal{O}_\rho(\mathfrak{U})$'s are objects of our main interest here.

Since $\mathfrak{U}$ is a Banach Lie group, and $\mathcal{L}(\mathcal{H})_s$ is (isomorphic to) its Lie algebra [3], the orbits $\mathcal{O}_\rho(\mathfrak{U})$ are the coadjoint orbits of $\mathfrak{U}$ through $\rho \in \mathfrak{T}_s \subset \mathcal{L}(\mathcal{H})_s^*$ ($\equiv$the topological dual of the Lie algebra $\mathcal{L}(\mathcal{H})_s$).

As I have learned from a discussion with colleagues Anatol Odzijewicz and Tudor Ratiu, there is an "innocently looking" problem connected also with coadjoint action of Lie groups, which is far not trivial in the general case. It is the question in which way the homogeneous spaces G/G_ρ of a Lie group G with their natural analytic manifold structure (with G_ρ being the stability subgroup of G at the point ρ), specifically their coadjoint orbits, are included into the topological spaces where the group acts. In more specific terms the problem is, whether the injective inclusion is an immersion and homeomorphism of the analytic manifold G/G_ρ onto a submanifold of the space $\mathcal{T}$ on which the group G acts. For example, an orbit $\mathcal{O}$ of a specific action of $\mathbb{R}$ on the two-torus $T^2 = S^1 \times S^1$, given by $\mathcal{O} := \{(\mathrm{e}^{\mathrm{i}t\omega_1}; \mathrm{e}^{\mathrm{i}t\omega_2}) : t \in \mathbb{R}\} \subset T^2$ with irrational quotient ω_1/ω_2, covers the torus densely, hence it is not a submanifold of T^2. As it is shown in a Kirillov's example [4] (cited and reproduced in [5, 14.1(f), p. 449]), such a pathologically looking case is possible also in the cases of finite-dimensional *coadjoint orbits*. There are also other possibilities for injectively immersed manifolds of not being submanifolds of the "ambient" space, cf. [5, p. 126] for an illustration. The more one could expect an occurrence of such phenomena in the case of infinite-dimensional orbits of Banach Lie groups.

Let $\mathcal{O}_\rho(\mathfrak{U}) = \mathfrak{U}/\mathfrak{U}_\rho$ be the homogeneous space of the unitary group $\mathfrak{U}$ of the infinite-dimensional Hilbert space $\mathcal{H}$ corresponding to an orbit of the action $u \mapsto u\rho u^*$, $u \in \mathfrak{U}$, on the space $\mathfrak{T}_s(\ni \rho)$ of symmetric trace-class operators in $\mathcal{L}(\mathcal{H})$ ($\mathfrak{U}_\rho$ is the stability subgroup of $\mathfrak{U}$ at ρ, namely $\mathfrak{U}_\rho := \{v \in \mathfrak{U} : v\rho v^* = \rho\}$). In the paper [1], the topology of the orbits $\mathcal{O}_\rho(\mathfrak{U})$, as well as the topology of their natural injection into the dual B-space (containing the predual $\mathfrak{T}_s$) were investigated. (Let us note here that a far reaching generalizations of some of structures developed and investigated in [1] are contained in [6]. That work was also stimulating for the here reported research.) It was proved in [1,6] (cf. [1, Proposition 2.1.5], and also [6, Theorem 7.5, Examples 7.9 and 7.10]), that orbits trough symmetric trace-class operators are injectively immersed into $\mathfrak{T}_s$ iff they are going trough operators with finite range. There was not completed, however, the proof of *regularity* of this embedding (in the terminology of Choquet-Bruhat et al. [7]) of such "finite-range" orbits. One of the aims of this paper is to fill this gap. Proving this in Theorem 2.5, it will be shown that the above mentioned "pathologies of embeddings" for the class of orbits $\mathcal{O}_\rho(\mathfrak{U})$ consisting of finite-range operators cannot occur. Proposition 3.1 shows that a certain types of "pathologies" are excluded also for orbits going through infinite-range trace class operators. Some additional related facts can be found in [8].

Let us note, that the posed question of whether the orbit is also a submanifold of the "ambient" space in which the group acts is easily and positively answered in the case of finite-dimensional Hilbert space $\mathcal{H}$. In that case the group $\mathfrak{U}$ is compact, so that the orbits are also compact and a continuous bijection of any compact space into a Hausdorff space is a closed mapping, hence a homeomorphism. For an infinite-dimensional $\mathcal{H}$, however, the orbits $\mathcal{O}_\rho(\mathfrak{U})$ are noncompact.

2. A proof of regularity of the embedding

We shall accept here some results from [1], mainly from Proposition 2.1.5 and Theorem 2.1.19; cf. also [6] for more general versions of the needed assertions. Some of the constructions formulated in this paper and connected with the proof of Theorem 2.5 might be, perhaps, also of independent interest, cf. also [8].

Let us describe first in more detail a formulation of the problem, and our strategy to approach it. It is known [3, Proposition 37, Chapter III, Section 3] that the unitary group $\mathfrak{U}$ of the W^*-algebra $\mathcal{L}(\mathcal{H})$ of all bounded operators on a complex Hilbert space $\mathcal{H}$ is a Banach Lie group, and its Lie algebra consists of all antisymmetric bounded linear operators $i\mathcal{L}(\mathcal{H})_s$, which is B-space isomorphic to $\mathcal{L}(\mathcal{H})_s$. The adjoint representation of $\mathfrak{U}$ in the B-space $\mathcal{L}(\mathcal{H})_s$ is $\mathrm{Ad} : \mathfrak{U} \to \mathcal{L}(\mathcal{L}(\mathcal{H})_s)$, $u \mapsto \mathrm{Ad}(u)$, with $\mathrm{Ad}(u)B := uBu^*\ \forall B \in \mathcal{L}(\mathcal{H})_s$. The representation we are here mostly interested in is the *coadjoint representation* of $\mathfrak{U}$ consisting of the transposed mappings $\mathrm{Ad}^*(u) := \mathrm{Ad}(u^{-1})^*$ to $\mathrm{Ad}(u^{-1})$'s, hence acting on continuous linear functionals $\nu \in \mathcal{L}(\mathcal{H})_s^*$, $\nu : \mathcal{L}(\mathcal{H})_s \to \mathbb{C}$, $B \mapsto \langle \nu; B\rangle$; the mapping $\mathrm{Ad}^*(u) : \mathcal{L}(\mathcal{H})_s^* \to \mathcal{L}(\mathcal{H})_s^*$ is determined by $\langle \mathrm{Ad}^*(u)\nu; B\rangle := \langle \nu; \mathrm{Ad}(u^{-1})B\rangle$. The subset of symmetric *normal* linear functionals is Ad^*-invariant, and it can be identified with the B-space $\mathfrak{T}_s \subset \mathcal{L}(\mathcal{H})_s^*$ of symmetric trace-class operators: $\nu(\in \mathfrak{T}_s) : B \mapsto \langle \nu; B\rangle := \mathrm{Tr}(\nu B)$; the space $\mathfrak{T}_s$ is a Banach space with the trace-norm $\|\nu\|_1 := \mathrm{Tr}|\nu|$, with the absolute value of the operator ν defined as the operator $|\nu| := \sqrt{\nu^*\nu} \in \mathcal{L}(\mathcal{H})$.

We are interested in comparison of two topologies introduced on the orbits $\mathcal{O}_\rho(\mathfrak{U}) := \{\mathrm{Ad}^*(u)\rho \equiv u\rho u^* : u^{-1} = u^* \in \mathfrak{U} \subset \mathcal{L}(\mathcal{H})\}$ of the coadjoint representation. Let us denote $\mathfrak{U}_\rho := \{u \in \mathfrak{U} : u\rho = \rho u\}$ $(\rho \in \mathfrak{T}_s)$. Then $\mathfrak{U}_\rho$ is a Lie subgroup of $\mathfrak{U}$ (cf. [1, Lemma 2.1.2], or [6, Proposition 6.8 and Theorem 7.5]), and the factor-space $\mathfrak{U}/\mathfrak{U}_\rho$ (which can be canonically identified, as a set, with $\mathcal{O}_\rho(\mathfrak{U})$) endowed with the factor-topology of the analytic Banach Lie group $\mathfrak{U}$ is an analytic Banach manifold [3, III.1.6, Proposition 11].

On the other side, the orbit $\mathcal{O}_\rho(\mathfrak{U})$ is naturally a subset of the Banach space $\mathfrak{T}_s$ endowed with the norm-topology given by the trace-norm $\|\cdot\|_1$. The topology induced on $\mathcal{O}_\rho(\mathfrak{U})$ from this B-space topology on $\mathfrak{T}_s$ need not coincide with the analytic manifold topology of

$\mathfrak{U}/\mathfrak{U}_\rho$. It is known that this coincidence is *not* the case for any ρ with infinite-dimensional range, cf. [1, Proposition 2.1.5] or [6, Example 7.9 and next Remark]. The coincidence of these two topologies means that the immersed subset $\iota(\mathfrak{U}/\mathfrak{U}_\rho) = \mathcal{O}_\rho(\mathfrak{U})$ of $\mathfrak{T}_s$ endowed with the topology of $\mathfrak{U}/\mathfrak{U}_\rho$ is a submanifold of $\mathfrak{T}_s$, or equivalently, that the inclusion mapping $\iota : \mathfrak{U}/\mathfrak{U}_\rho \to \mathfrak{T}_s$ (provided that ι is immersion) is a homeomorphism of $\mathfrak{U}/\mathfrak{U}_\rho$ onto the topological subspace $\mathcal{O}_\rho(\mathfrak{U}) \subset \mathfrak{T}_s$ [9, 5.8.3].

We intend to prove that, for any $\rho = \rho^* \in \mathfrak{F}$ (:= the linear space of finite-rank operators in a complex Hilbert space $\mathcal{H}$), the topology induced on the subset $\mathcal{O}_\rho(\mathfrak{U}) := \{u\rho u^* : u^{-1} = u^* \in \mathfrak{U} \subset \mathcal{L}(\mathcal{H})\}$ from the overlying (resp. "ambient") Banach space of symmetric trace-class operators $\mathfrak{T}_s$ is equivalent to the topology of the set $\mathcal{O}_\rho(\mathfrak{U})$ considered as the factor-space $\mathfrak{U}/\mathfrak{U}_\rho$. If the inclusion $\iota : \mathfrak{U}/\mathfrak{U}_\rho \to \mathcal{O}_\rho(\mathfrak{U}) \subset \mathfrak{T}_s$, $[u]_\rho \mapsto \iota([u]_\rho) := u\rho u^*$, where $[u]_\rho := \{v \in \mathfrak{U} : v\rho v^* = u\rho u^*\}$, is an (injective) immersion, and if it were also homeomorphism of $\mathfrak{U}/\mathfrak{U}_\rho$ onto $\iota(\mathfrak{U}/\mathfrak{U}_\rho) = \mathcal{O}_\rho(\mathfrak{U})$, then $\mathcal{O}_\rho(\mathfrak{U})$ would be a submanifold of $\mathfrak{T}_s$, cf. [9, 5.8.3].

Let us sketch our "strategy" of proving this claim here. It was proved in [1, Proposition 2.1.5] (cf. also [6, Corollary 7.8 and Example 7.9]) that $\mathcal{O}_\rho(\mathfrak{U})$ is an immersed submanifold (i.e. the injective inclusion $\iota : \mathfrak{U}/\mathfrak{U}_\rho \to \mathcal{O}_\rho(\mathfrak{U}) \subset \mathfrak{T}_s$ is an immersion [9, 5.7.1]) of $\mathfrak{T}_s$ for $\dim(\rho) := \mathrm{rank}(\rho) < \infty$. To obtain the wanted result, we are going to prove that the inverse mapping $\iota^{-1} : \mathcal{O}_\rho(\mathfrak{U}) \to \mathfrak{U}/\mathfrak{U}_\rho$ is also continuous. It will be useful to our technique to use the metric-space expression of continuity of mappings, i.e. the "$\epsilon \leftrightarrow \delta$ language". It is useful to realize for this that the considered homogeneous spaces $\mathfrak{U}/\mathfrak{U}_\rho$ are all (for $\dim(\rho) < \infty$) Riemann manifolds endowed with strong Riemannian metrics [1, Theorem 2.1.19]. Then the manifold topology is given by the corresponding distance function [10, Proposition 4.64], hence all the considered topologies are metric ones, i.e. the metric on $\mathfrak{U}$ given by the operator norm $\|u - v\|$, the Riemannian topology on $\mathcal{O}_\rho(\mathfrak{U})$ represented by a distance function $d_\rho(\rho', u\rho' u^*)$ (the distance d_ρ will not be explicitly calculated here, we shall not need it), and also the topology of the space $\mathfrak{T}_s$, into which $\mathcal{O}_\rho(\mathfrak{U})$ is embedded, is given by the norm-distance $\|\rho' - u\rho' u^*\|_1$. [We shall also use the same notation for elements $\rho' \in \mathcal{O}_\rho(\mathfrak{U}) \subset \mathfrak{T}_s$, and for their images $\iota^{-1}(\rho') \equiv \rho' \in \mathfrak{U}/\mathfrak{U}_\rho$.]

We have to prove that, for any $\rho' \in \mathcal{O}_\rho(\mathfrak{U})$, and for an arbitrary $\epsilon' > 0$ there is a $\delta' > 0$ such that if there is any element $\rho'' \in \mathcal{O}_\rho(\mathfrak{U})$ with $\|\rho'' - \rho'\|_1 < \delta'$, then it is also $d_\rho(\rho', \rho'') < \epsilon'$. The projection $\Pi_\rho : \mathfrak{U} \to \mathfrak{U}/\mathfrak{U}_\rho$, $u \mapsto [u]_\rho \approx u\rho u^*$ is continuous (here $[u]_\rho \approx u\rho u^*$ means the canonical identification of the left cosets $[u]_\rho \subset \mathfrak{U}$ with their realization as the points $u\rho u^*$ of $\mathcal{O}_\rho(\mathfrak{U})$). We can use this continuity to avoid necessity of (possibly complicated) calculation of explicit forms of d_ρ (cf. Proposition 3.2). Since Π_ρ is continuous, to any $\rho' \in \mathfrak{U}/\mathfrak{U}_\rho$, and to any $\epsilon' > 0$ there is an $\epsilon > 0$ such that if $\|I_\mathcal{H} - v\| < \epsilon$, then also $d_\rho(\rho', v\rho' v^*) < \epsilon'$. So, if we could find to any $\epsilon > 0$, and to any given $\rho' \in \mathcal{O}_\rho(\mathfrak{U})$ such a $\delta' > 0$ that for each $\rho'' := u\rho' u^* : \|\rho' - \rho''\|_1 < \delta'$ *it is possible to find* a unitary v such that also $\rho'' = v\rho' v^*$, and simultaneously $\|I_\mathcal{H} - v\| < \epsilon$, then the continuity of ι^{-1} in the (arbitrarily chosen) point $\rho' \in \mathcal{O}_\rho(\mathfrak{U}) \subset \mathfrak{T}_s$ will be proved. This will be the wanted result. *It will be proved then the continuity of* $\iota^{-1} : \mathcal{O}_\rho(\mathfrak{U}) \to \mathfrak{U}/\mathfrak{U}_\rho$ *on whole its domain* $\mathcal{O}_\rho(\mathfrak{U})$.

We shall proceed essentially in the just indicated way, but to avoid explicit calculation of dependence $\epsilon \mapsto \delta'(\epsilon)$, we shall use also another known continuity, namely the continuous dependence of the spectral projections $F_j : \rho'' \mapsto F_j(\rho'')$ of $\rho'' := \sum_j \lambda_j F_j \in \mathcal{O}_\rho(\mathfrak{U})$ on the ρ'' itself, cf. Lemma 2.2.

The following lemma provides us with a 'freedom' in dealing with various topologies induced on the considered orbits.

Lemma 2.1. *The topologies coming from the trace-class B-space* $L^1(\mathcal{H})$ $(:= \mathfrak{T}(\mathcal{H}) \supset \mathfrak{T}_s \supset \mathfrak{F}_N)$, *from the Hilbert–Schmidt B-space* $L^2(\mathcal{H})$ $(:= \mathfrak{H} \supset \mathfrak{H}_s \supset \mathfrak{T}_s \supset \mathfrak{F}_N)$, *as well as from the* C^**-algebra of all bounded operators* $L^\infty(\mathcal{H}) := \mathcal{L}(\mathcal{H})$ $(\supset \mathcal{L}(\mathcal{H})_s \supset \mathfrak{H}_s \supset \mathfrak{T}_s \supset \mathfrak{F}_N)$, *induced on the subset of symmetric finite-range operators* $\mathfrak{F}_N$ *with a fixed maximal dimension N of their ranges are all equivalent.*

Proof. Let N be maximal dimension of ranges of the considered operators $A, B \in \mathfrak{F}_N$, $A = A^*$, $B = B^*$, hence the ranges of the operators $A - B$ are of maximal dimension $2N$. The considered topologies are all metric topologies induced on $\mathfrak{F}_N$ by the corresponding norms from the "above lying" spaces. The distances between A and B are correspondingly given by $\|A-B\|_1 := \mathrm{Tr}|A-B|$, $\|A-B\|_2 := \sqrt{\mathrm{Tr}|A-B|^2}$, and $\|A-B\| =: \|A-B\|_\infty =$ the maximal eigenvalue of $|A-B|$, where $|A-B|$ denotes the absolute value of the operator $A-B$, $|A-B| := \sqrt{(A-B)^*(A-B)}$. Generally, it is $\|C\|_\infty \equiv \|C\| \le \|C\|_2 \le \|C\|_1$ for any trace-class operator C. Conversely, also due to the mentioned inequalities, one clearly has $\|A-B\|_2 \le \|A-B\|_1 \le 2N\|A-B\|_\infty \le 2N\|A-B\|_2$ for $A, B \in \mathfrak{F}_N$. This shows that all the three metric topologies are on $\mathfrak{F}_N \subset \mathcal{L}(\mathcal{H})$ mutually equivalent. □

We shall need a rather indirect, but a quite "faithful" expression for "proximity" of finite-range operators on the same orbit considered as a subset of the B-space $\mathfrak{T}_s$, which would be more difficult to express directly with a help of the usual norms of their differences. To this end we shall need the following lemma.

Lemma 2.2. *Let us consider the subset* $\mathfrak{B}_\sigma$ *of bounded operators* $\mathcal{L}(\mathcal{H})$ *consisting of all bounded symmetric operators* $\rho \in \mathcal{L}(\mathcal{H})$ *with a given purely discrete finite spectrum* $\sigma := \{\lambda_0, \lambda_1, \lambda_2, \dots, \lambda_n\} \subset \mathbb{C}$. *Their spectral projections* $F_j \equiv F_j(\rho)$ $(j = 0, 1, 2, \dots, n)$ *are continuous functions of* $\rho \in \mathfrak{B}_\sigma$:

$$\rho := \sum_{j=0}^{n} \lambda_j \cdot F_j,$$

in the operator norm topology of $\mathcal{L}(\mathcal{H})$.

Proof. The spectral projections of any symmetric operator ρ are uniquely determined by that operator, hence for a given discrete spectrum (e.g. $\rho \in \mathfrak{B}_\sigma$) the projections corresponding to fixed spectral values are uniquely determined functions of the operators $\rho \in \mathfrak{B}_\sigma$. By the use of a spectral functional calculus one can choose some functions $p_j : \mathbb{R} \to \mathbb{R}$ such that $p_j(\lambda_k) \equiv \delta_{jk}$. Then $p_j(\rho) = F_j := F_j(\rho)\,\forall j$. Let us choose for the functions p_j polynomials; we define for any complex $z \in \mathbb{C}$

$$p_j(z) := \prod_{k(\neq j)=0}^{n} \frac{z - \lambda_k}{\lambda_j - \lambda_k}, \tag{2.1}$$

what gives $p_j(\rho) = F_j(\rho)$, and the continuity of $\rho \mapsto F_j(\rho)$ on (any subset of) $\mathfrak{B}_\sigma$ is explicitly seen. □

These two lemmas lead immediately to the following corollary.

Corollary 2.3. *The spectral projections F_j of finite-range operators $\rho \in \mathfrak{B}_\sigma \cap \mathfrak{F}_N$ are* (*on the set $\mathfrak{B}_\sigma \cap \mathfrak{F}_N$*) *continuous functions $\rho \mapsto F_j(\rho)$ of these operators in any of the considered* (*i.e. trace, Hilbert–Schmidt, and $\mathcal{L}(\mathcal{H})$*) *topologies* (*taken independently on the domain-, or range-sides*).

We shall use in the following text also the Dirac notation for vectors and operators in a complex Hilbert space: $|x\rangle := x \in \mathcal{H}$ will denote a vector, $\langle x|y\rangle$ is the scalar product of such vectors (linear in the *second* factor), and $|x\rangle\langle y|$ the operator of one-dimensional range such that $|x\rangle\langle y| : \sum_j c_j|z_j\rangle \mapsto |x\rangle\langle y| \cdot \sum_j c_j|z_j\rangle := (\sum_j c_j\langle y|z_j\rangle)|x\rangle$.

The constructions needed in the proof of the main theorem use also a more detailed description of consequences of "proximity" of two projections described in the following lemma.

Lemma 2.4. *Let E, F be two orthogonal projections of finite-dimensional ranges of equal dimensions $N := \dim E = \dim F := \mathrm{Tr}(E)$ in an infinite-dimensional Hilbert space $\mathcal{H}$.*

Assume that $E \wedge F = 0$, i.e. the subspaces $\mathcal{E} := E\mathcal{H}$ and $\mathcal{F} := F\mathcal{H}$ have no nonzero common vectors. Let us also denote $\mathcal{E} \vee \mathcal{F} := \mathcal{E} + \mathcal{F} = (E \vee F)\mathcal{H}$ the $2N$-dimensional linear hull in $\mathcal{H}$ of $\mathcal{E} \cup \mathcal{F}$. Let

$$\mathrm{Tr}[(E - F)^2] \equiv \|E - F\|_2^2 < 2. \tag{2.2}$$

Then:

(i) *For any one-dimensional projections given by normalized vectors $e \in \mathcal{E}$, $f \in \mathcal{F}$: $|e\rangle\langle e| =: P_e \le E$ (i.e., $P_e \cdot E = P_e$), and $|f\rangle\langle f| =: P_f \le F$, it is $P_e \cdot F \neq 0$, and $P_f \cdot E \neq 0$.*

(ii) *There exists an orthonormal basis $\{e_j : j = 1, 2, \dots, N := \dim E\} \subset \mathcal{H}$ in $\mathcal{E}$, i.e. $\sum_j P_{e_j} = E$, such that one can find an orthonormal basis of $\mathcal{F} : \{f_j : j = 1, 2, \dots, N\} \subset \mathcal{H}$ (i.e. $\sum_j P_{f_j} = F$), satisfying the relations*

$$P_{f_j}(E - P_{e_j}) = 0, \qquad P_{e_j}(F - P_{f_j}) = 0 \quad \forall j. \tag{2.3}$$

(iii) *Point* (ii) *means that these orthonormal systems $(e_j : j = 1, 2, \dots, N\}$, and $\{f_j : j = 1, 2, \dots, N\}$, decomposing E and F, are in a certain strong sense mutually "affiliated"*:

$$F|e_j\rangle = |f_j\rangle\langle f_j|e_j\rangle, \quad \forall j = 1, 2, \dots, N, \quad 0 \neq \langle f_j|e_j\rangle \in \mathbb{C},$$
$$\langle e_j|e_j\rangle \equiv 1 \equiv \langle f_j|f_j\rangle, \tag{2.4}$$

i.e. from a specific orthonormal 'decomposition' $\{e_j : j = 1, 2, \dots, N\}$ of $\mathcal{E}$ the orthonormal system $\{f_j : j = 1, 2, \dots, N\}$ 'decomposing' $\mathcal{F}$ and satisfying (2.3) *is obtained, uniquely up to nonzero numerical factors, simply by element-wise orthogonal projections of e_j's onto $\mathcal{F} := F\mathcal{H}$.*

(iv) *The above mentioned specific orthonormal basis* $\{e_j : j = 1, 2, \dots, N\}$ *determines also* (*up to* 'phase factors') *an orthonormal basis* $\{e_j^\perp : j = 1, 2, \dots, N\}$ *of* $\mathcal{E}^\perp := [(E \vee F) - E]\mathcal{H} = (\mathcal{E} \vee \mathcal{F}) \ominus \mathcal{E}$, *such that* $f_j = \alpha_j e_j + \beta_j e_j^\perp$, $\alpha_j \cdot \beta_j \neq 0\ (\forall j)$.

Proof.

(i) Let there be a projection $P_e \leq E$ such that $P_e F = 0$. Let $e_1 := e$, and let $\{e_j : j = 1, 2, \dots, N\}$ *be an orthonormal system decomposing* E, $E = \sum_{j=1}^N P_{e_j}$. Then

$$\mathrm{Tr}(EF) = \mathrm{Tr}[(E - P_e)F] = \sum_{j=2}^{N} \mathrm{Tr}(P_{e_j}F) \leq N - 1, \tag{2.5}$$

since always it is $\mathrm{Tr}(P_x F) \leq 1\, \forall x \in \mathcal{H}$. The estimate (2.5) would be then in contradiction with the assumption (2.2), since $\mathrm{Tr}[(E - F)^2] = 2(N - \mathrm{Tr}(EF))$. Due to the symmetry of the assumed conditions with respect to the exchange $E \leftrightarrow F$, one obtains also $P_f E \neq 0$. This implies validity of (i).

(ii) We have to prove existence of the bases $\{e_j\} := \{e_j : j = 1, 2, \dots, N := \dim E\}$, and $\{f_j : j = 1, 2, \dots, N = \dim F\}$ of $\mathcal{E}$, resp. $\mathcal{F}$ satisfying (2.3).

This means to find an orthonormal basis $\{e_j : j = 1, 2, \dots, N\}$ of $\mathcal{E}$ such that its element-wise projections onto $\mathcal{F}$ are proportional to f_j's, cf. (2.4). This also means that for such a basis $\{e_j\} \subset \mathcal{E}$ the projections $F|e_j\rangle \in \mathcal{F}$ are nonzero and mutually orthogonal.

Statement (i) ensures that all the projections $F|e\rangle$ of all nonzero vectors $e \in \mathcal{E}$ are nonzero, i.e. the restriction $EFE \in \mathcal{L}(\mathcal{E})$ of the projector F to the subspace $\mathcal{E} \subset \mathcal{H}$ has trivial kernel: $\mathrm{Ker}_{\mathcal{E}}(EFE) = 0$. This implies that the bounded operator $EFE = (FE)^* FE$ on $\mathcal{E}$ is strictly positive and there is an orthonormal basis $\{e_j\}$ of $\mathcal{E}$ in which the matrix $\langle e_j|EFE|e_k\rangle = \langle e_j|F|e_k\rangle$ is diagonal, with strictly positive diagonal elements $\|Fe_j\|^2$.

Let us define then, e.g., $f_j := \|Fe_j\|^{-1} \cdot Fe_j$, $j = 1, 2, \dots, N$; these elements form the wanted basis of $\mathcal{F}$, resp. specify the decomposition of the projector F satisfying together with the just found basis $\{e_j\}$ the relations (2.3). This proves (ii).

(iii) That statement is just a rephrasing of (ii); the uniqueness also is seen from (2.3).

(iv) Since each $f_j \in \mathcal{F}$ constructed as above is orthogonal to all the e_k $(k \neq j)$, and $\langle f_j|e_j\rangle \neq 0$, but it is also $E^\perp f_j \neq 0$ (since $f_j \notin \mathcal{E}$), with $E^\perp := E \vee F - E$, f_j is expressible in the form

$$f_j := \alpha_j e_j + \beta_j e_j^\perp \quad \forall j, \tag{2.6}$$

where $e_j^\perp \in \mathcal{E}^\perp := E^\perp \mathcal{H}$ is some normalized vector determined by f_j up to a 'phase factor', e.g. $e_j^\perp := \|E^\perp f_j\|^{-1} E^\perp f_j$.

We also see that all $\alpha_j \cdot \beta_j \neq 0$, since all $f_j \notin \mathcal{E}$, but also $f_j \notin \mathcal{E}^\perp$.

The orthogonality between the vectors f_j's: $\langle f_j|f_k\rangle \equiv \delta_{jk}$ implies also the orthogonality relations for $e_j^\perp$'s: $\langle e_j^\perp|e_k^\perp\rangle = \delta_{jk}$. □

We are prepared now to prove the regularity of embeddings into $\mathfrak{T}_s$ of unitary orbits through finite-range symmetric operators.

Theorem 2.5. *Let $0 \neq \rho = \rho^* \in \mathfrak{F}$ (:= the set of all finite-range operators on $\mathcal{H}$), $\mathcal{O}_\rho(\mathfrak{U}) := \{u\rho u^* : u \in \mathfrak{U}\} \subset \mathfrak{T}_s$. The unitary orbit $\mathcal{O}_\rho(\mathfrak{U})$ is a regularly embedded [7, p. 550] submanifold of the Banach space $\mathfrak{T}_s$ of symmetric trace-class operators endowed with its trace norm, i.e. the injection $\iota : \mathfrak{U}/\mathfrak{U}_\rho \to \mathcal{O}_\rho(\mathfrak{U}) \subset \mathfrak{T}_s$, $[u]_\rho \mapsto u\rho u^* \in \mathfrak{T}_s$* is a homeomorphism.

Proof. The mapping $\Pi_\rho : \mathfrak{U} \to \mathfrak{U}/\mathfrak{U}_\rho$, $u \mapsto [u]_\rho(\approx u\rho u^*) \in \mathfrak{U}/\mathfrak{U}_\rho$ is an analytic submersion [3, Section III.1.6 and Proposition 11], and the inclusion $\iota : \mathfrak{U}/\mathfrak{U}_\rho \to \mathfrak{T}_s$ is an injective immersion (cf. [1, Proposition 2.1.5] or also [6, Examples 7.9 and 7.10] in a more general setting; also formulations and proofs of other statements cited here from Bóna [1] could be found also in [6] in some modified and/or generalized forms), hence the composition $\iota \circ \Pi_\rho : \mathfrak{U} \to \mathfrak{T}_s$ is continuous. We want to prove, that the inverse (identity) mapping $\iota^{-1} : \mathcal{O}_\rho(\mathfrak{U})(\subset \mathfrak{T}_s) \to \mathfrak{U}/\mathfrak{U}_\rho$ is also continuous, if the domain $\mathcal{O}_\rho(\mathfrak{U})$ of ι^{-1} is taken in the relative topology of the corresponding "ambient" space $\mathfrak{T}_s \subset L^1(\mathcal{H})$. It suffices to prove the wanted continuity in each point ρ of the orbit $\mathcal{O}_\rho(\mathfrak{U})$.

Our strategy (sketched in the beginning of this section) is as follows. Let us choose any $\rho \in \mathcal{O}_\rho(\mathfrak{U}) \subset \mathfrak{T}_s$, and let its (identical) copy in the topological space $\mathfrak{U}/\mathfrak{U}_\rho$ be denoted by $\iota^{-1}(\rho)$. We shall show that to any $\epsilon' > 0$ there is a $\delta' > 0$ such that if $\|\rho - u\rho u^*\|_1 < \delta'$ for some $u \in \mathfrak{U}$, then also $d_\rho(\iota^{-1}(\rho), \iota^{-1}(u\rho u^*)) < \epsilon'$, what is the elementary definition of continuity of the mapping ι^{-1} from $\mathcal{O}_\rho(\mathfrak{U})$ (with the induced topology from $\mathfrak{T}_s$) onto the analytic manifold $\mathfrak{U}/\mathfrak{U}_\rho$ ($\equiv \mathcal{O}_\rho(\mathfrak{U})$, as a set) in a point ρ. This will be shown in two steps, i.e. by

- (i) using the continuity of the projection $\Pi_\rho : \mathfrak{U} \to \mathcal{O}_\rho(\mathfrak{U})$, $u \mapsto u\rho u^*$ for a choice of $\epsilon > 0$ (for the given $\epsilon' > 0$) such, that if $\|v - I_\mathcal{H}\| < \epsilon$ for $v \in \mathfrak{U}$, then also $d_\rho(\iota^{-1}(\rho), \iota^{-1}(v\rho v^*)) < \epsilon'$, and
- (ii) showing that there exists, to this $\epsilon > 0$, a $\delta' > 0$ such, that if there is some element $\rho' = u\rho u^* \in \mathcal{O}_\rho(\mathfrak{U})$ in the δ'-neighborhood of ρ in the space $\mathfrak{T}_s : \|\rho - u\rho u^*\|_1 < \delta'$, then *it is possible to find* (to each such u separately, without any additional requirements to the mapping $u \mapsto v \equiv v(u)$, or to the mapping $\rho' \mapsto v(\rho') \equiv v$) a unitary $v \in \mathfrak{U} : \|v - I_\mathcal{H}\| < \epsilon$, such that $v\rho v^* = u\rho u^*$.

The proof will be direct: *A* construction of a unitary $v : \|v - I_\mathcal{H}\| < \epsilon$ for any given $\rho' = u\rho u^*$ lying "sufficiently close" to ρ in $\mathfrak{T}_s$ (i.e. $\|\rho - u\rho u^*\|_1 < \delta'$), such that it is also $\rho' = v\rho v^*$.

Let us write $\rho = \sum_{j=1}^n \lambda_j E_j$, $0 < n < \infty$, where $\lambda_j \neq \lambda_k$ for $j \neq k$, E_j are the orthogonal projections of the spectral measure of $\rho = \rho^*$, $0 < \dim E_j := \mathrm{Tr}(E_j) =: N_j < \infty$ $(\forall j \neq 0)$, $E_0 := I_\mathcal{H} - \sum_{j=1}^n E_j =: I_\mathcal{H} - E$, $\lambda_0 := 0$, $\sum_{j=1}^n N_j =: N$. Let us denote $F_j := uE_ju^*$ $(\forall j)$, hence $\rho' := u\rho u^* = \sum_j \lambda_j F_j$, and also let $F := \sum_{j=1}^n F_j$.

It is clear that the nonnegative numbers $N_j - \mathrm{Tr}(E_jF_j(\rho'))$ and $N - \mathrm{Tr}(EF(\rho'))$ are all continuous functions of ρ' and for $\rho' = \rho$ they all are zero. This can be seen, e.g. by representing the projection operators $F_j \equiv F_j(\rho')$ by polynomials p_j of the operators ρ', as it was done in Lemma 2.2, cf. also Corollary 2.3.

These considerations imply that, for any given $0 < \delta^\circ < 1, 0 < \delta_j^\circ < 1$ $(j = 1, 2, \ldots, n)$, for all sufficiently small $\delta' > 0$, and for all such $\rho' = u\rho u^*$ that $\|\rho - u\rho u^*\|_1 < \delta'$, one obtains

$$\begin{aligned} &0 \le N_j - \mathrm{Tr}(E_j F_j(\rho')) =: \delta_j < \delta_j^\circ < 1, \quad j = 1, 2, \dots, n, \\ &0 \le N - \mathrm{Tr}(EF(\rho')) =: \delta < \delta^\circ < 1, \end{aligned} \tag{2.7}$$

where δ, δ_j ($j = 1, 2, \dots, n$) can be chosen arbitrarily small positive numbers (i.e. they can be bounded from above by arbitrarily small positive upper bounds $\delta^\circ, \delta_j^\circ$ ($j = 1, 2, \dots, n$) determining the choice of the mentioned $\delta' > 0$, what is possible due to the continuous dependence on ρ' of the expressions entering into (2.7)).

Let us choose now $0 < \epsilon < 1$, and assume that the above mentioned δ' is such that for any of the considered ρ' it is

$$\delta \le \sum_{j=1}^{n} \delta_j < \frac{\epsilon^2}{4}, \tag{2.8}$$

where the first inequality is a consequence of the definitions (2.7). Let us note that we need not here any explicit expression for the dependence $\epsilon \mapsto \delta' \equiv \delta'(\epsilon)$; it could be 'in principle' obtained, however, from explicit formulas for the functions $\rho' \mapsto F_j(\rho')$, e.g. from those given in the proof of Lemma 2.2.

We shall construct now, for any $\rho' = u\rho u^*$ with $\|\rho' - \rho\|_1 < \delta'$, such a unitary $v \in \mathfrak{U}$, that $v\rho v^* = u\rho u^*$, and simultaneously $\|v - I_{\mathcal{H}}\| < \epsilon$.

Let us denote $Q_j := E_j \wedge F_j$, $E'_j := E_j - Q_j$, $F'_j := F_j - Q_j$, $Q := E \wedge F$, $E' := E - Q$, $F' := F - Q$, $E'^{\perp} := (E' \vee F') - E' = E \vee F - E$, $F'^{\perp} := (E' \vee F') - F' = E \vee F - F$, $N'_j := \dim E_j - \dim Q_j = \dim E'_j = \dim F'_j$, $N' := \dim E - \dim Q = \dim E' = \dim F' = \dim E'^{\perp} = \dim F'^{\perp}$. Observe that $(E-F)^2 = [(E \vee F - E) - (E \vee F - F)]^2 = (E'^{\perp} - F'^{\perp})^2$. Also it is $\mathrm{Tr}(EF) = \mathrm{Tr}(E'F' + Q) = \mathrm{Tr}(E'F') + N - N'$, and $\dim(E \vee F) = N + N'$. So that we obtain

$$\mathrm{Tr}[(E - F)^2] = 2[N - \mathrm{Tr}(EF)] = \mathrm{Tr}[(E'^{\perp} - F'^{\perp})^2] = 2[N' - \mathrm{Tr}(E'^{\perp} F'^{\perp})]. \tag{2.9}$$

Now we can apply Lemma 2.4 separately to each of the couples of projections

$$(E'_j; F'_j), \quad j = 1, 2, \dots, n, \qquad (E'^{\perp}; F'^{\perp}), \tag{2.10}$$

and construct the orthonormal systems $\{e_k^{(j)} : k = 1, 2, \dots, N'_j\}$ forming the convenient bases of every $\mathcal{E}'_j := E'_j \mathcal{H}$ ($j = 1, 2, \dots, n$), and also the basis $\{e_k^{\perp} : k = 1, 2, \dots, N'\}$ of $\mathcal{E}'^{\perp} := E'^{\perp}\mathcal{H}$, such that their respective orthogonal projections onto the spaces $\mathcal{F}'_j := F'_j \mathcal{H}$ ($j = 1, 2, \dots, n$), and $\mathcal{F}'^{\perp} := F'^{\perp}\mathcal{H}$, corresponding to the second projection in the considered pair of (2.10), are the orthogonal (and afterwards normalized) bases $\{f_k^{(j)} : k = 1, 2, \dots, N'_j\}$ of $\mathcal{F}'_j$ ($j = 1, 2, \dots, n$), and the orthonormal basis $\{f_k^{\perp} : k = 1, 2, \dots, N'\}$ of $\mathcal{F}'^{\perp}$. Let us choose any orthonormal bases $\{e_k^{(j)} \equiv f_k^{(j)} : k = N'_j + 1, \dots, N_j\}$ of all the subspaces $\mathcal{Q}_j := Q_j \mathcal{H}$, $j = 1, 2, \dots, n$. We have obtained in this way two orthonormal systems $\{e_k^{(j)}, e_i^{\perp} : k = 1, 2, \dots, N_j, j = 1, 2, \dots, n, i = 1, 2, \dots, N'\}$, and $\{f_k^{(j)}, f_i^{\perp} : k = 1, 2, \dots, N_j, j = 1, 2, \dots, n, i = 1, 2, \dots, N'\}$, each forming a basis of the subspace

$\mathcal{E} \vee \mathcal{F} := (E \vee F)\mathcal{H}$. Remember also the "cross-orthogonality" of the mutually "affiliated" orthonormal systems:

$$\langle f_k^{(j)} | e_l^{(j)} \rangle = 0, \quad (j = 1, 2, \ldots, n), \qquad \langle f_k^{\perp} | e_l^{\perp} \rangle = 0; \quad \text{for } l \neq k \ \forall k, l. \tag{2.11}$$

Let also the arbitrary phase factors at the all f's entering into the orthonormal sets be chosen so that for all possible values of the indices it is

$$\langle f_l^{\perp} | e_l^{\perp} \rangle > 0, \qquad \langle f_k^{(j)} | e_k^{(j)} \rangle > 0. \tag{2.12}$$

Now we shall define the wanted unitary v. Let the restriction of v to $\mathcal{H} \ominus (\mathcal{E} \vee \mathcal{F}) := (\mathcal{E} \vee \mathcal{F})^{\perp}$ be the identity (i.e. $v\rceil_{\mathcal{H}\ominus(\mathcal{E}\vee\mathcal{F})} := I_{\mathcal{H}\ominus(\mathcal{E}\vee\mathcal{F})}$), and its restriction to $\mathcal{E} \vee \mathcal{F}$ is defined as the linear transformation between the constructed orthonormal systems forming two bases in $\mathcal{E} \vee \mathcal{F}$ specified by

$$v e_k^{(j)} := f_k^{(j)}, \qquad v e_i^{\perp} := f_i^{\perp}; \quad \forall i, j, k. \tag{2.13}$$

It is clear from this definition of v, esp. from (2.13) that $\sum_{j=1}^{n} \lambda_j F_j = v(\sum_{j=1}^{n} \lambda_j E_j)v^*$, i.e. $\rho' = v\rho v^*$. Let us show next that $\|v - I_{\mathcal{H}}\| < \epsilon$. Since $(v - I_{\mathcal{H}})\rceil_{\mathcal{H}\ominus(\mathcal{E}\vee\mathcal{F})} = 0$, we shall estimate the Hilbert–Schmidt norm of $(v - I_{\mathcal{H}})$ in the subspace $\mathcal{E} \vee \mathcal{F}$. Let $\mathrm{Tr}'(C)$ be the trace of the restriction of $C \in \mathcal{L}(\mathcal{H})$ to $\mathcal{E} \vee \mathcal{F}$. We obtain with a help of (2.12):

$$\begin{aligned}
\|v - I_{\mathcal{H}}\|_2^2 &= \mathrm{Tr}'(2I_{\mathcal{H}} - v - v^*) \\
&= 2(N + N') - 2\sum_{j=1}^{n}\sum_{k=1}^{N_j} \langle f_k^{(j)} | e_k^{(j)} \rangle - 2\sum_{j=1}^{N'} \langle f_j^{\perp} | e_j^{\perp} \rangle \\
&= 2\sum_{j=1}^{n}\left[N_j - \sum_{k=1}^{N_j} \langle f_k^{(j)} | e_k^{(j)} \rangle\right] + 2\left[N' - \sum_{j=1}^{N'} \langle f_j^{\perp} | e_j^{\perp} \rangle\right] \\
&\leq 2\sum_{j=1}^{n}\left[N_j - \sum_{k=1}^{N_j} |\langle f_k^{(j)} | e_k^{(j)} \rangle|^2\right] + 2\left[N' - \sum_{j=1}^{N'} |\langle f_j^{\perp} | e_j^{\perp} \rangle|^2\right] \\
&= 2\sum_{j=1}^{n}[N_j - \mathrm{Tr}(E_j F_j)] + 2[N' - \mathrm{Tr}(E'^{\perp} F'^{\perp})] \\
&= 2\sum_{j=1}^{n}[N_j - \mathrm{Tr}(E_j F_j)] + 2[N - \mathrm{Tr}(EF)],
\end{aligned} \tag{2.14}$$

where we have used again the orthogonality properties (2.11) of the vectors inside each "block" corresponding to E_j, $j = 1, 2, \ldots, n$, as well as to $E'^{\perp} : \sum_{j=1}^{n} E_j + E'^{\perp} = E \vee F$; it was also used the fact that $|\langle f|e \rangle|^2 \leq |\langle f|e \rangle|$ for any normalized vectors $e, f \in \mathcal{H}$, as well as the relation (2.9).

Now we shall use the definitions (2.7), and the assumption (2.8). We obtain

$$\|v - I_{\mathcal{H}}\|^2 \leq \|v - I_{\mathcal{H}}\|_2^2 \leq 2\sum_{j=1}^{n} \delta_j + 2\delta \leq 4\sum_{j=1}^{n} \delta_j < \epsilon^2, \tag{2.15}$$

what is the desired result. □

Hence, each orbit of the coadjoint action of $\mathfrak{U}$ going through density matrices ρ with only finite number of different eigenvalues is a submanifold of $\mathfrak{T}_s$. There is an open neighborhood of any point ν of $\mathfrak{U}/\mathfrak{U}_\rho$ which coincides with the intersection of the embedded $\mathcal{O}_\rho(\mathfrak{U})$ into $\mathfrak{T}_s$ with an open neighborhood of the point ν in $\mathfrak{T}_s$.

Another possibility of proving this theorem is indicated in the next section, where such a proof for the specific case of $\mathcal{O}_\rho(\mathfrak{U}) := P(\mathcal{H})$ is given.

3. Some other related results

To prove the promised closeness of the unitary coadjoint orbit going through any symmetric trace-class operator, we shall use an encoding of the spectral invariants (i.e. the spectra, and their multiplicities) of these operators into finite positive measures on $\mathbb{R}$.

Proposition 3.1. *The unitary orbits $\mathcal{O}_\rho(\mathfrak{U})$ (for any $\rho \in \mathfrak{T}_s$) are closed subsets of $\mathfrak{T}_s$.*

Proof. Let us take now the smooth (although differentiability will not be exploited here) numerical functions $a_n : \rho \mapsto a_n(\rho) := \mathrm{Tr}(\rho^{n+2})$ determined for all symmetric trace-class operators $\rho \in \mathfrak{T}_s$. The numbers $a_n(\nu)$ are constant on $\mathcal{O}_\rho(\mathfrak{U}) : a_n(u\rho u^*) \equiv a_n(\rho)\, \forall u \in \mathfrak{U}$, $\rho \in \mathfrak{T}_s$. It is claimed that fixing the infinite sequence $\{a_n(\rho), n = 0, 1, 2, \dots\}$ of real numbers one can determine the unitary orbit $\mathcal{O}_\rho(\mathfrak{U}) \subset \mathfrak{T}_s$ uniquely. This can be seen as follows: the orbit is determined by the spectral invariants of any $\nu \in \mathcal{O}_\rho(\mathfrak{U})$, i.e. by its nonzero eigenvalues and their (finite) multiplicities. These might be, however, determined by a measure μ_ρ on $\mathbb{R}$, namely the (not normalized) measure given by the characteristic function $t(\in \mathbb{R}) \mapsto \mathrm{Tr}(\rho^2\, \mathrm{e}^{\mathrm{i}t\rho})$, the moments of which are exactly the numbers $a_n(\rho)$. That measure expressed by the nonzero eigenvalues λ_j of ρ, and their multiplicities m_j, has the form

$$\mu_\rho = \sum_j \lambda_j^2 \cdot m_j \cdot \delta_{\lambda_j}, \tag{3.1}$$

where δ_λ is the Dirac probabilistic measure concentrated in the point λ. It is clear that this measure μ_ρ determines the orbit uniquely. The uniqueness of the solution of the Hamburger problem of moments (see [11, Theorem X.4, and Example 4 in Chapter X.6]) for the moments given by the sequence $\{a_n(\rho), n = 0, 1, 2, \dots\}$ proves that the measure μ_ρ is in turn determined by the sequence $\{a_n(\rho)\}$ uniquely.

Since the functions $\rho \mapsto a_n(\rho)$ are continuous in the trace-norm topology, the intersection of the (closed) inverse images $a_n^{-1}[a_n(\rho)]$ $(n \in \mathbb{Z}_+)$ is

$$\mathcal{O}_\rho(\mathfrak{U}) = \bigcap_{n=0}^{\infty} \{\nu \in \mathfrak{T}_s : a_n(\nu) = a_n(\rho)\}, \tag{3.2}$$

what is a closed subset of $\mathfrak{T}_s$ in this topology. □

Next will be given an independent way of proving Theorem 2.5, but only for a specific case of the orbit $\mathcal{O}_\rho(\mathfrak{U})$ with $\rho = P_x$, i.e. for the projective Hilbert space $P(\mathcal{H})$. A use of that method for other orbits $\mathcal{O}_\rho(\mathfrak{U})$ would need calculation of the distance functions $d_\rho(u\rho u^*, v\rho v^*)$ on the Riemannian manifolds $\mathcal{O}_\rho(\mathfrak{U})$ for a general ρ of finite-range; for $\rho \in \mathfrak{T}_s$ with infinite range the claim of Theorem 2.5 is false (cf. [1]).

Proposition 3.2. *The unitary orbit $\mathcal{O}_\rho(\mathfrak{U})$ going through a one-dimensional projection $\rho := P_x$ $(0 \neq x \in \mathcal{H})$ is a submanifold of (i.e. it is regularly embedded into) the space $\mathfrak{T}_s$ of symmetric trace-class operators.*

Proof. It is known that the Riemannian distance function on $P(\mathcal{H})$ is

$$d(P_x, P_y) = \sqrt{2}\arccos\sqrt{\mathrm{Tr}(P_x P_y)}. \tag{3.3}$$

(The derivation of the distance $d(P_x, P_y)$ is easy after accepting the (plausible looking) assumption that any geodesic is contained in the submanifold of $P(\mathcal{H})$ homeomorphic to a real two-dimensional sphere representing the projective Hilbert space of the two-dimensional complex subspace of $\mathcal{H}$ spanned by $\{x, y\}$. The nontrivial part of the proof consists in justification of this assumption [12].)

On the other hand, the distance between the same projections in the "ambient space" $\mathfrak{T}_s$ is

$$\mathrm{Tr}|P_x - P_y| = 2[1 - \mathrm{Tr}(P_x P_y)]^{1/2}, \tag{3.4}$$

what is easily obtained as the sum $|\lambda_1| + |\lambda_2|$ of absolute values of the two nonzero real eigenvalues (if $P_x \neq P_y$) of $P_x - P_y$. Since $\mathrm{Tr}(P_x - P_y) = \lambda_1 + \lambda_2 = 0$, one has $\lambda_1 = -\lambda_2 =: \lambda > 0$. Because $2\lambda^2 = \mathrm{Tr}[(P_x - P_y)^2] = 2[1 - \mathrm{Tr}(P_x P_y)]$, one obtains $\lambda = \sqrt{1 - \mathrm{Tr}(P_x P_y)}$, hence the result (3.4). We see that these two metrics are mutually equivalent.

This implies that the convergence of some sequence $\{P_{y_n} : n \in \mathbb{Z}_+\}$ of points of this orbit to a chosen point $P_x \in \mathcal{O}_{P_x}(\mathfrak{U})$ *in the space* $\mathfrak{T}_s$ means also its convergence *on the homogeneous space* $\mathfrak{U}/\mathfrak{U}_{P_x}$, what gives the wanted continuity of the inverse ι^{-1} of the injective immersion (it was proved earlier in [1] that ι is an immersion) $\iota : \mathfrak{U}/\mathfrak{U}_{P_x} \to \mathcal{O}_{P_x}(\mathfrak{U}) = P(\mathcal{H}) \subset \mathfrak{T}_s$ (the last set $P(\mathcal{H})$ is taken in the relative topology of $\mathfrak{T}_s$). This means that the injection ι is a homeomorphism, hence $P(\mathcal{H})$ is a submanifold (cf. [9]) of $\mathfrak{T}_s$. □

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